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The Boltzmann collision integrals for a combination of Maxwellians

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The gain and loss integrals in the Boltzmann equation for a rigid sphere gas are evaluated in closed form for a distribution which can be expressed as a linear combination of Maxwellians. Application to the Mott-Smith bimodal distribution shows that the gain is also bimodal, but the two modes in the gain are less pronounced than in the distribution. Implications of these results for simple collision models in non-equilibrium flow are discussed.

1. Introduction

The collision terms in the Boltzmann equation are represented by integrals which are defined for any two given distribution functions. There are many situations in which it is useful to know these integrals when both distributions are Maxwellian, but with different parameters (the collision terms are zero when the parameters are the same). One such situation is the study of the slow mutual equilibration of different species in a gas mixture, when each species attains a (different) Maxwellian distribution by self-collisions. Another, to be discussed in detail in the following paper (Narasimha & Deshpande 1969), arises when the distribution for a single gas is represented (possibly in some approximation) as a linear combination of Maxwellians.

We show here (in § 2) that for a rigid sphere gas these integrals can be expressed in closed form. Apart from the applications mentioned above, this result is of much intrinsic interest because it provides insight into the structure of the collision integrals in a situation involving large departure from equilibrium. In § 3 we discuss the implications of the present results for an assessment of collision models of the BGK type; a similar evaluation of Monte-Carlo computations of the collision integrals will be reported later.

2. Analysis

The Boltzmann collision integrals for any two distribution functions f_i, f_j can be written in general as

$$\mathscr{J}(f_i,f_j) = \mathscr{G}(f_i,f_j) - f_i\mathscr{L}(f_j), \tag{2.1a}$$

where the gain and loss terms are respectively given by

$$\mathscr{G}(f_i,f_j) = \int f_i(\mathbf{v}')f_j(\mathbf{w}')\,gb\,db\,d\epsilon\,D\mathbf{w}, \tag{2.1b}$$

$$\mathscr{L}(f_j) = \int f_j(\mathbf{w})\,gb\,db\,d\epsilon\,D\mathbf{w}. \tag{2.1c}$$

Here $\mathbf{v}$ is the velocity at which the integrals are being evaluated, $\mathbf{w}$ is the velocity of the collision partner, $\mathbf{g} = \mathbf{v}-\mathbf{w}$ is the relative velocity, b is the impact parameter, ϵ is the azimuthal angle and primes denote values after collision (see Chapman & Cowling 1960). We use $D\mathbf{w}$ to denote a volume element in $\mathbf{w}$-space.

In the following, we shall represent the Maxwellian distribution by

$$F_i(\mathbf{v}) = n_i(\beta_i/\pi)^{\frac{3}{2}}\exp\{-\beta_i(\mathbf{v}-\mathbf{u}_i)^2\}, \tag{2.2}$$

where the parameters n_i, $\mathbf{u}_i$ and β_i are respectively the number density, gas velocity and the inverse square of the most probable thermal speed for the distribution. Further, we shall, for brevity, often denote the gain and loss integrals for Maxwellians by appropriate subscripts

$$\mathscr{G}_{ij} \equiv \mathscr{G}(F_i,F_j), \quad \mathscr{L}_j \equiv \mathscr{L}(F_j), \tag{2.3}$$

etc. The subscripts may refer to different components of a mixture of gases, or to different terms in an expansion of f into Maxwellians for a single gas.† In either case, the total gain or loss will be given by a series whose typical terms are proportional to the quantities defined by (2.3). We shall for this reason find it convenient to phrase most of the following analysis for the single gas; but at any stage it is easy to interpret the results for a mixture.

For rigid spheres of diameter σ,‡ the loss term $\mathscr{L}_j$ is well known; it is in fact calculated by Chapman & Cowling (1960, p. 93) in their analysis of the dependence of collision frequency on speed:

$$\mathscr{L}(F_j) = n_j\sigma^2\left(\frac{\pi}{\beta_j}\right)^{\frac{1}{2}}\left[\exp(-\mathscr{C}_j^2)+\frac{1+2\mathscr{C}_j^2}{\mathscr{C}_j}\operatorname{erf}\mathscr{C}_j\right], \tag{2.4a}$$

where
$$\mathscr{C}_j = \beta_j^{\frac{1}{2}}(\mathbf{v}-\mathbf{u}_j), \quad \operatorname{erf} z = \int_0^z e^{-t^2}\,dt. \tag{2.4b}$$

This result can be rewritten in a concise and, as we shall see later, more revealing form by utilizing the relation between the error function and the confluent hypergeometric function Φ. Making use of standard transformations and recurrence relations for Φ (Erdélyi *et al.* 1953), this alternative expression for $\mathscr{L}(F_j)$ is

$$\mathscr{L}(F_j) = 2\sigma^2(\pi/\beta_j)^2F_j\,\Phi(2;\tfrac{3}{2};\mathscr{C}_j^2) = \mathscr{G}(F_j,F_j)/F_j, \tag{2.4c}$$

where the second equality follows from the vanishing of the collision integrals $\mathscr{J}(F_j,F_j)$ for a Maxwellian F_j.

Proceeding to the calculation of the gain term $\mathscr{G}_{ij}$, we first non-dimensionalize

† How best to make such an expansion in the particular problem of shock structure is the subject of the following paper.

‡ In a mixture of gases, σ represents the mean diameter of the colliding molecules, i, j.

all velocities with respect to $\beta_i^{\frac{1}{2}}$. Using the dynamical relations governing a collision between two particles, we can write the velocities after collision as

$$\mathbf{v}' = \mathbf{v} - \mathbf{k}g\cos\psi, \quad \mathbf{w}' = \mathbf{v} - \mathbf{g} + \mathbf{k}g\cos\psi, \tag{2.5}$$

where $\mathbf{k}$ is the unit vector along the apse-line and ψ is its colatitude with respect to $\mathbf{g}$ (the co-ordinate systems we shall find convenient to adopt in velocity space are shown in figure 1). It must be remembered that ψ depends only on b and g in general, and only on b for rigid spheres:

$$b = \sigma \sin\psi.$$

Also $\mathbf{k}$ depends only on ψ, ϵ and the direction of $\mathbf{g}$.

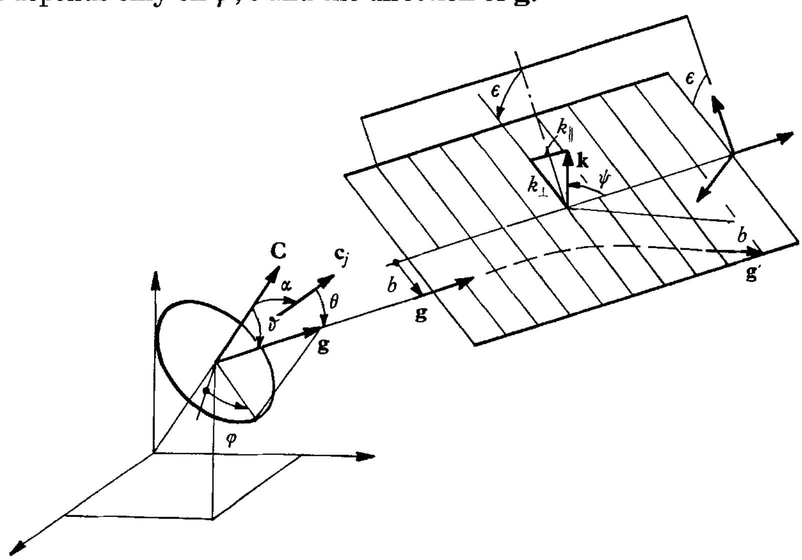

FIGURE 1. Relative orientation of different co-ordinate systems in velocity space.

Using (2.5), we obtain

$$F_i(\mathbf{v}')\,F_j(\mathbf{w}') = F_i(\mathbf{v})\,F_j(\mathbf{w})\exp\{2\mathbf{k}\,.\,\mathbf{C}g\cos\psi + 2\beta_{ji}\mathbf{c}_j\,.\,\mathbf{g} - g^2(\cos^2\psi + \beta_{ji}\sin^2\psi)\}, \tag{2.6}$$

where $$\mathbf{c}_j = \mathbf{v} - \mathbf{u}_j, \quad \mathbf{C} = \mathbf{c}_i - \beta_j\mathbf{c}_j, \quad \beta_{ji} \equiv \beta_j/\beta_i. \tag{2.7}$$

In (2.6), ϵ appears only in the term $\mathbf{k}\,.\,\mathbf{C}$; in fact if we resolve $\mathbf{k}$ and $\mathbf{C}$ along and normal to $\mathbf{g}$ and denote the components by the subscripts $\parallel$ and $\perp$, only $\mathbf{k}_\perp\,.\,\mathbf{C}_\perp$ involves ϵ (in the form of a linear combination of $\sin\epsilon$ and $\cos\epsilon$). The integration with respect to ϵ in (2.1*b*) is therefore easily performed, using the result

$$\int_0^{2\pi} d\epsilon \exp(\mathbf{k}\,.\,\mathbf{C}\,2g\cos\psi) = 2\pi\exp(2gC_\parallel\cos^2\psi)\,I_0(gC_\perp\sin 2\psi), \tag{2.8}$$

where I_0 is the modified Bessel function of the first kind and zeroth-order.†

† For the many relations involving special functions which we shall have occasion to use here, we will not always give a specific reference if the result can be found in volume 2 of the well-known book by Erdélyi *et al.* (1953). A considerable amount of analysis is however involved in the work, and readers interested in obtaining a more general and detailed account will find it in Narasimha & Deshpande (1968) and Deshpande & Narasimha (1969).

Further, as $\mathbf{v}$ is fixed during the integration in (2.1*b*), the volume element $D\mathbf{w}$ can be replaced by the element

$$D\mathbf{g} = g^2 \sin\vartheta \, d\vartheta \, d\varphi \, dg,$$

adopting a spherical polar co-ordinate system around $\mathbf{C}$ (figure 1). Now φ appears only in the term $\mathbf{c}_j \cdot \mathbf{g}$ in (2.6), and by exactly the same argument as in obtaining (2.8), we have

$$\int_0^{2\pi} d\varphi \exp(2\beta_j \mathbf{c}_j \cdot \mathbf{g}) = 2\pi \exp(2\beta_{ji} g c_j \cos\vartheta \cos\alpha) \, I_0(2\beta_{ji} g c_j \sin\vartheta \sin\alpha), \quad (2.9)$$

where α is the polar angle of $\mathbf{c}_j$. Putting (2.8) and (2.9) into (2.1*b*) gives a product of the I_0 functions, from which the terms in $\sin\vartheta$ may be separated by the convenient expansion

$$I_0(gC \sin\vartheta \sin 2\psi) \, I_0(2\beta_{ji} g c_j \sin\vartheta \sin\alpha) = \sum_{m=0}^{\infty} \frac{(\beta_{ji} c_j g \sin\alpha \sin\vartheta)^{2m}}{m!^2} \, {}_2F_1\left(-m, -m; 1; \frac{C^2 \sin^2 2\psi}{2\beta_{ji}^2 c_j^2 \sin^2\alpha}\right), \quad (2.10)$$

where ${}_2F_1$ is the usual hypergeometric function, and we have replaced $C_\perp$ in (2.8) by $C \sin\vartheta$. Now we perform integrations with respect to ϑ, noting that as $C_\parallel = C\cos\vartheta$ the exponentials in (2.8) and (2.9) both contain $\cos\vartheta$, and that each term in the series (2.10) is therefore of the form

$$\int_0^{\pi} d\vartheta \exp(-Z\cos\vartheta)(\sin\vartheta)^{2m+1} = m! \, \pi^{\frac{1}{2}} (\tfrac{1}{2}Z)^{-m-\frac{1}{2}} \, I_{m+\frac{1}{2}}(Z) = 2m! (\tfrac{1}{2}Z)^{-m} \, i_m(Z), \quad (2.11)$$

where $i_m(Z)$ is the modified spherical Bessel function of order m.

The net result of all these integrations is that we can write $\mathscr{G}(F_i, F_j)$ as the infinite series

$$\mathscr{G}_{ij} = \frac{8\pi^2}{\beta_i^2} F_i F_j \int dg \, b \, db \exp\{-g^2(\cos^2\psi + \beta_{ji}\sin^2\psi)\} \times \sum_{m=0}^{\infty} \frac{(\beta_{ji} c_j \sin\alpha)^{2m}}{m! \, \chi^m} \, {}_2F_1\left(-m, -m; 1; \frac{C^2 \sin^2 2\psi}{4\beta_{ji}^2 c_j^2 \sin^2\alpha}\right) \times i_m(2\chi g) \, g^{m+3}, \quad (2.12)$$

where $$\chi \equiv C\cos^2\psi + \beta_{ji} c_j \cos\alpha.$$

Up to this point, no assumption has been made about the molecular model. If we assume rigid spheres, b can be replaced by $\sigma \sin\psi$, and the integration with respect to g in (2.12) can be completed, using the relation

$$\int_0^{\infty} dt \, i_m(\alpha t) \, t^{\rho-1} \exp(-\gamma^2 t^2) = (\tfrac{1}{8}\pi)^{\frac{1}{2}} (\alpha/2\gamma)^{m+\frac{1}{2}} \gamma^{-\frac{1}{2}} \frac{\Gamma\{\frac{1}{2}(m+\rho)\}}{\Gamma(m+\frac{3}{2})} \, \Phi\left(\frac{m+\rho}{2}; m+\tfrac{3}{2}; \frac{\alpha^2}{4\gamma^2}\right). \quad (2.13)$$

Finally, we note the identity

$${}_2F_1(-m, -m; 1; \Psi) = \frac{1}{2\pi} \int_0^{2\pi} d\theta (1 + \Psi^{\frac{1}{2}} e^{-i\theta})^m (1 + \Psi^{\frac{1}{2}} e^{i\theta})^m,$$

and use the summation formula (Slater 1960)

$$\Phi(a;b;x+y) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{y^n}{n!} \Phi(a+n;b+n;x).$$

Introducing
$$\zeta^{-1} = 1+\beta_{ji}\tan^2\psi, \tag{2.14}$$

we can write (2.12), after some algebra, as the double integral

$$\mathscr{G}_{ij} = \frac{2\pi\sigma^2}{\beta_i\beta_j} F_i F_j \int_0^{\pi} d\theta \int_0^1 d\zeta \times \Phi\{2;\tfrac{3}{2};\zeta\mathscr{C}_i^2+(1-\zeta)\mathscr{C}_j^2+2\zeta^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}}|\mathscr{C}_i\times\mathscr{C}_j|\cos\theta\}. \tag{2.15}$$

When $i = j$ this clearly reduces to the well-known result for a single Maxwellian, already quoted in (2.4). For $i \neq j$, (2.15) expresses the gain as a weighted integral of the result for a single Maxwellian and further shows that $\mathscr{G}_{ij}$ is symmetric:

$$\mathscr{G}(F_i, F_j) = \mathscr{G}(F_j, F_i). \tag{2.16}$$

The double integral in (2.15) can be evaluated in closed form as shown in the appendix. Taking (A 6) with the expression (2.4) for the loss term, we obtain for the total collision integral

$$\mathscr{J}(F_i, F_j) = \frac{\pi^2\sigma^2}{\beta_i\beta_j} F_i F_j \left[\frac{1}{2R}\{\Phi(1;\tfrac{1}{2};Q+R)-\Phi(1;\tfrac{1}{2};Q-R)\} - 2\frac{\beta_i}{\beta_j}\Phi(2;\tfrac{3}{2};\mathscr{C}_j^2)\right], \tag{2.17}$$

where
$$Q \equiv \tfrac{1}{2}(\mathscr{C}_i^2+\mathscr{C}_j^2), \quad R^2 = \tfrac{1}{4}(\mathscr{C}_i^2-\mathscr{C}_j^2)^2+|\mathscr{C}_i\times\mathscr{C}_j|^2; \tag{2.18}$$

the first two terms in (2.17) give the gain $\mathscr{G}_{ij}$ and the last term gives the loss $F_i\mathscr{L}_j$.

3. Results and discussion

From (2.17) we see that to obtain the gain and the loss integrals it is only necessary to calculate the relevant hypergeometric function Φ. A simple computer program has been written for doing this, using the well-known series representation when the argument is less than 10, and an asymptotic expansion when the argument is larger (Erdélyi *et al.* 1953).

Using this program the gain and loss terms have been computed in a few different cases, for a Mott-Smith type distribution (Mott-Smith 1951),

$$f_0 = (1-\nu)F_1+\nu F_2, \tag{3.1}$$

where F_1 and F_2 are the Maxwellian distributions corresponding to the equilibrium states of a monatomic gas respectively on the cold and hot side of a normal shock. Typical results for a flow Mach number $M_1 = 10{\cdot}0$, $\nu = \frac{1}{2}$ are shown in figures 2 and 3; the distribution (3.1) at these conditions has also been plotted for comparison in figure 2. The most interesting feature of the results is the presence of two maxima in the gain, as in the distribution itself, although the supersonic peak in the gain is less pronounced. Thus the gain operator smooths the peakiness of f to an appreciable extent, but not as much as to make it a completely symmetric

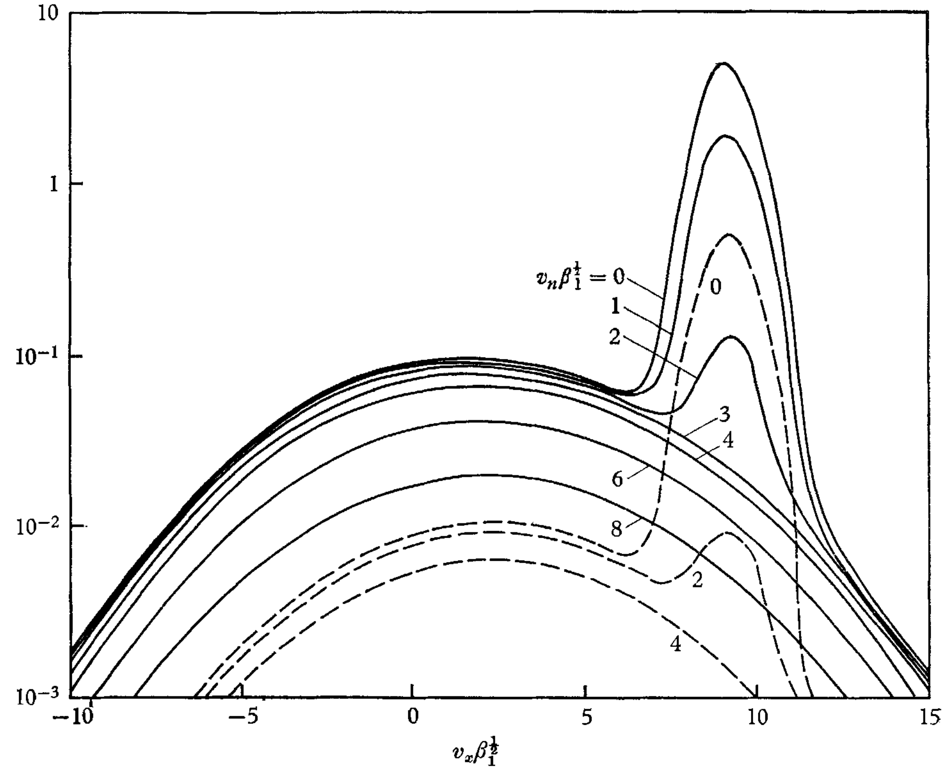

FIGURE 2. The distribution function and the loss term for a Mott-Smith ansatz with $\nu = \frac{1}{2}$ at $M_1 = 10\cdot0$. The curves are labelled by the values of $\beta_1^{\frac12} v_n$, where v_n is the velocity component normal to v_x. ———, $f_0\mathscr{L}(f_0)/\beta_1 n_1^2\sigma^2$; — — — —, $f_0/n_1(\beta_1/\pi)^{\frac32}$.

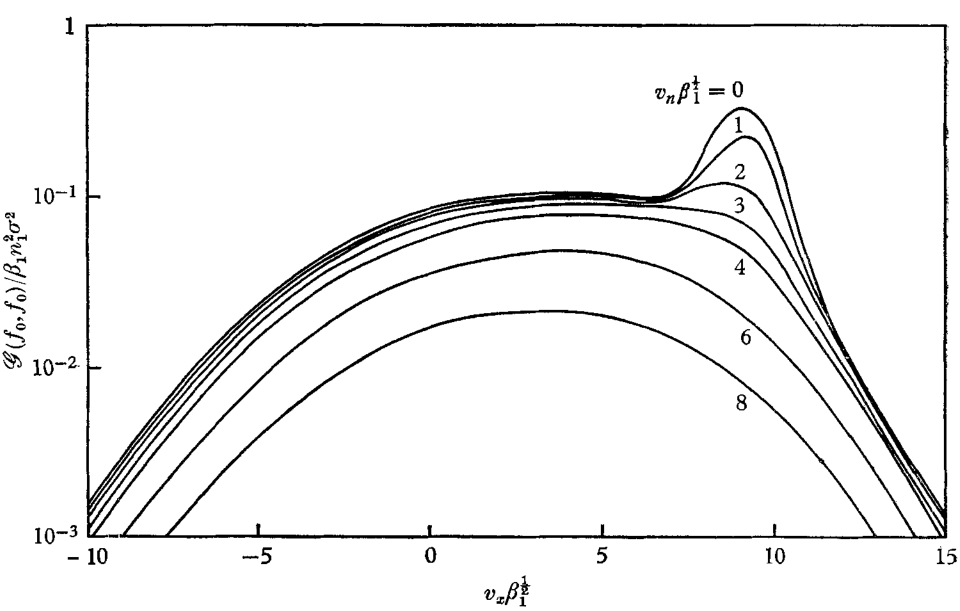

FIGURE 3. The gain term for the same distribution as in figure 2.

Maxwellian, as assumed in the BGK model (Bhatnagar, Gross & Krook 1954). The ratio of the maxima in $\mathscr{G}$ and in f are shown in figure 4 as a function of the shock Mach number M_1.

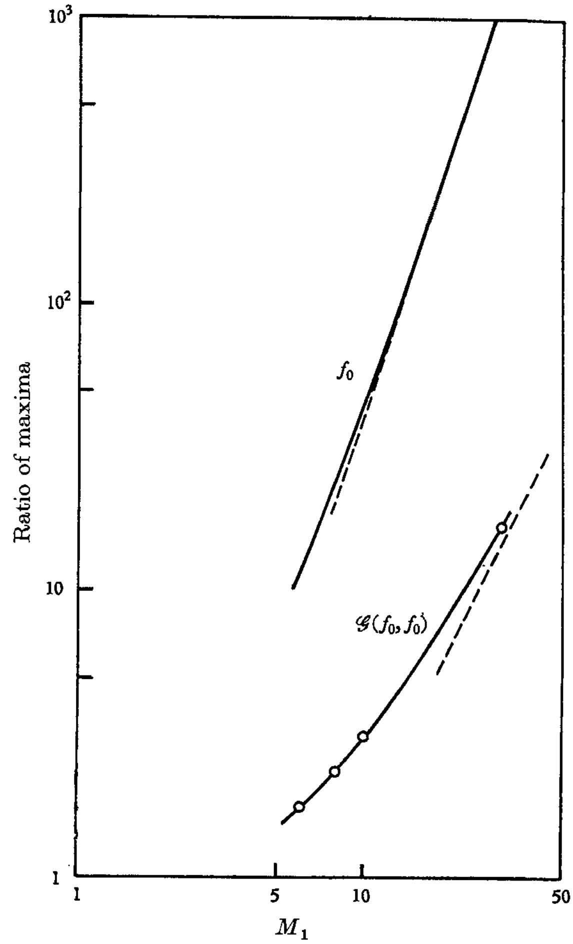

FIGURE 4. Ratio of the two maximum values of the distribution and the gain. Same conditions as in figure 2. — — — —, asymptotic, $M_1 \to \infty$.

It is of interest to pursue this question a little further by studying the asymptotic behaviour of $\mathscr{G}$ and $\mathscr{L}$ when $M_1 \to \infty$. In this limit we might expect the collision terms to exhibit, like f itself, different behaviours near the supersonic peak and elsewhere. We therefore consider respectively an 'inner' limit defined as the process $\mathscr{C}_1$ fixed, $M_1 \to \infty$ (to describe the region near the supersonic peak) and an 'outer' limit defined by $\mathscr{C}_2$ fixed, $\mathscr{C}_1 \to \infty$, $M_1 \to \infty$ (to describe the rest of velocity space).

For the distribution (3.1), the total gain is

$$\mathscr{G}(f_0, f_0) = (1-\nu)^2 \mathscr{G}_{11} + 2\nu(1-\nu)\, \mathscr{G}_{12} + \nu^2 \mathscr{G}_{22}. \tag{3.2}$$

In the inner limit $\mathscr{C}_2$ takes the constant value

$$\mathscr{C}_2 = \beta_2^{\frac{1}{2}}(\mathbf{u}_1 - \mathbf{u}_2) \equiv \mathscr{C}_{21}, \tag{3.3}$$

say, and it is easily shown from (2.17) that

$$\mathscr{G} \approx \frac{2n_1^2\beta_1\sigma^2}{\pi}\bigg[\Phi(2,\tfrac{3}{2},\mathscr{C}_1^2)\exp(-2\mathscr{C}_1^2)+\frac{\beta^{\frac{1}{2}}}{\epsilon R}\{\Phi(1,\tfrac{1}{2},Q+R)$$
$$-\Phi(1,\tfrac{1}{2},Q-R)\}\exp(-\mathscr{C}_1^2-\mathscr{C}_{21}^2)+(\beta/\epsilon^2)\,\Phi(2,\tfrac{3}{2},\mathscr{C}_{21}^2)\exp(-2\mathscr{C}_1^2)\bigg], \quad (3.4)$$

where $\beta \equiv \beta_2/\beta_1 = O(M_1^{-2}), \quad \epsilon = n_1/n_2.$

The leading term in (3.4) comes from $\mathscr{G}_{11}$.

In the outer limit $\mathscr{C}_2 = O(1)$ and $\mathscr{C}_1 \to \infty$, and using well-known asymptotic expansions for the Φ one obtains the outer expansion of the gain as

$$\mathscr{G} \approx \frac{2\sigma^2 n_1^2\beta_1}{\pi}\left[\pi^{\frac{1}{2}}\frac{\beta}{\epsilon}\frac{\exp(-\mathscr{C}_2^2)}{|\mathscr{C}_2-\mathscr{C}_{21}|}+\frac{\beta}{\epsilon^2}\exp(-2\mathscr{C}_2^2)\,\Phi(2,\tfrac{3}{2},\mathscr{C}_2^2)\right]. \quad (3.5)$$

Comparing (3.4) and (3.5), we see that the ratio of the peaks in $\mathscr{G}$ is $O(\beta^{-1})$, whereas that in f is $O(\beta^{-\frac{3}{2}})$. Thus the gain operator reduces the peakiness by a factor of $O(\beta^{\frac{1}{2}})$. In the BGK model it is assumed that the gain is an isotropic Maxwellian with only one maximum; a crude but plausible argument in favour of this assumption is the isotropy of two-body scattering in a centre-of-mass system for rigid spheres (Liepmann, Narasimha & Chahine 1962). The present results demonstrate that the total gain cannot be isotropic, and indeed contains two peaks. However, it must be noted that as the inner solution covers a velocity-space volume of $O(\beta^{\frac{3}{2}})$ compared to the outer, the contribution of the inner peak in $\mathscr{G}$ to its integral over $\mathbf{v}$ is $O(\beta^{\frac{1}{2}})$ compared to that of the outer solution. Thus, in the limit $M_1 \to \infty$, F_1 tends to a delta function at $\mathbf{v} = \mathbf{u}_1$, whereas $\mathscr{G}$, although tending to infinity at this point, is weaker than a delta function and contributes nothing to the integral in the limit. Whatever success the BGK model can claim in describing the gain term must then be attributed to this fact.

The total collision term $\mathscr{J}$ shows a slightly different behaviour. Again using similar methods, we obtain

$$\mathscr{J} \approx \frac{n_1^2\sigma^2\beta_1}{\pi}\frac{2}{\epsilon\beta^{\frac{1}{2}}}\exp(-\mathscr{C}_1^2-\mathscr{C}_{21}^2)\bigg[-\Phi(2,\tfrac{3}{2},\mathscr{C}_{21}^2)$$
$$+\frac{\beta}{2R}\{\Phi(1,\tfrac{1}{2},Q+R)-\Phi(1,\tfrac{1}{2},Q-R)\}-\beta^2\Phi(2,\tfrac{3}{2},\mathscr{C}_1^2)\bigg] \quad (3.6)$$

in the inner limit, where the leading term is the loss at the supersonic peak, $F_1\mathscr{L}(F_2)$. The outer expression is

$$\mathscr{J} \approx \frac{2\sigma^2 n_1^2\beta_1}{\pi}\frac{\pi^{\frac{1}{2}}\beta}{2\epsilon}\exp(-\mathscr{C}_2^2)\left[\frac{1}{|\mathscr{C}_2-\mathscr{C}_{21}|}-\tfrac{1}{2}|\mathscr{C}_2-\mathscr{C}_{21}|\right]. \quad (3.7)$$

Comparison of (3.6) with (3.7) shows that the outer limit of $\mathscr{J}$ is $O(\beta^{\frac{3}{2}})$ times the inner limit, i.e. it has the same ordering as f itself, and both limits contribute equally to the integral.

The asymptotic dependence of the ratio of the maxima, as obtained from this analysis, is also shown in figure 4.

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Appendix

The most convenient way to evaluate the double integral in (2.15) appears to be to introduce certain new angular variables and apply the theory of spherical harmonics. We therefore put

$$\left.\begin{aligned} \zeta \equiv \cos^2 \tfrac{1}{2}\tau, \quad (\mathscr{C}_i^2 - \mathscr{C}_j^2)/2R \equiv \cos\omega, \\ \tfrac{1}{2}(\mathscr{C}_i^2 - \mathscr{C}_j^2)\cos\tau + |\mathscr{C}_i \times \mathscr{C}_j| \cos\theta \sin\tau \equiv R\cos\Theta, \end{aligned}\right\} \tag{A 1}$$

where R is the variable defined by (2.18) and

$$\cos\Theta = \cos\omega\cos\tau + \sin\omega\sin\tau\cos\theta. \tag{A 2}$$

We further replace the hypergeometric function Φ in (2.15) by the integral representation

$$\Phi(\alpha, \gamma, x) = \frac{1}{2\pi i}\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha+1)}{\Gamma(\alpha)}\int_c dt\, t^{\alpha-1}(t-1)^{\gamma-\alpha-1}e^{xt}, \tag{A 3}$$

where c is a double loop in the complex t-plane as described by Erdélyi *et al.* (1953, vol. 2, p. 272). The argument of the hypergeometric function, in the new variables introduced here and in (2.18), becomes $Q + R\cos\Theta$. Introducing (A 3) into (2.15) $\mathscr{G}_{ij}$ can be expressed as a triple integral, involving the term $\exp(Rt\cos\Theta)$:

$$\mathscr{G}_{ij} = -i(\tfrac{1}{2}!)^2\frac{F_iF_j}{\beta_i\beta_j}\int_0^\pi d\theta\int_0^\pi d\tau\sin\tau\int_c dt\,t(t-1)^{-\frac{3}{2}}\exp\left[t(Q+R\cos\Theta)\right]. \tag{A 4}$$

Now recalling (A 2), we can obtain from the theory of spherical harmonics the expansion

$$\exp(Rt\cos\Theta) = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n}(-)^m\frac{(n-m)!}{(n+m)!}\,i_n(Rt)\cos m\theta\,P_n^m(\cos\tau)\,P_n^m(\cos\omega), \tag{A 5}$$

where the P_n^m are the associated Legendre polynomials. The series (A 5) is analogous to the well-known expansion of a plane wave in spherical harmonics (e.g. Morse & Feshbach 1953, p. 1466). We now carry out successively the integrations in θ and τ in (A 4); using the familiar orthogonality relations for the P_n^m, it will be found that only the term corresponding to $n = 0$ in (A 5) makes a contribution

$$\int_0^\pi d\tau\sin\tau\int_0^\pi d\theta\exp(Rt\cos\Theta) = 2\pi i_0(Rt).$$

Putting this in (A 4) and noting that

$$i_0(Rt) = (Rt)^{-1}\sinh Rt,$$

we will obtain two integrals of the type (A 3). Expressing these back in terms of hypergeometric functions, we have the final result

$$\mathscr{G}_{ij} = \frac{\pi^2\sigma^2}{2R\beta_i\beta_j}F_iF_j[\Phi(1, \tfrac{1}{2}, Q+R) - \Phi(1, \tfrac{1}{2}, Q-R)]. \tag{A 6}$$

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